Scott and Swarup's regular neighbourhood as a tree of cylinders

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Abstract

Let G be a finitely presented group. Scott and Swarup have constructed a canonical splitting of G which encloses all almost invariant sets over virtually polycyclic subgroups of a given length. We give an alternative construction of this regular neighbourhood, by showing that it is the tree of cylinders of a JSJ splitting.

1 Introduction

Scott and Swarup have constructed in [SS03] a canonical graph of groups decomposition (or splitting) of a finitely presented group G, which encloses all almost invariant sets over virtually polycyclic subgroups of a given length n (VPC_n groups), in particular over virtually cyclic subgroups for $n = 1$.

Almost invariant sets generalize splittings: a splitting is analogous to an *embedded* codimension-one submanifold of a manifold M, while an almost invariant set is analogous to an *immersed* codimension-one submanifold.

Two splittings are *compatible* if they have a common refinement, in the sense that both can be obtained from the refinement by collapsing some edges. For example, two splittings induced by disjoint embedded codimension-one submanifolds are compatible.

Enclosing is a generalisation of this notion to almost invariant sets: in the analogy above, given two codimension-one submanifolds F_1, F_2 of M with F_1 immersed and F_2 embedded, F_1 is enclosed in a connected component of $M \setminus F_2$ if one can isotope F_1 into this component.

Scott and Swarup's construction is called the *regular neighbourhood* of all almost invariant sets over VPC_n subgroups. This is analogous to the topological regular neighbourhood of a finite union of (non-disjoint) im-

mersed codimension-one submanifolds: it defines a splitting which encloses the initial submanifolds.

One main virtue of their splitting is the fact that it is canonical: it is invariant under automorphisms of G. Because of this, it is often quite different from usual JSJ splittings, which are unique only up to deformation: the canonical object is the JSJ deformation space [For03, GLa].

The main reason for this rigidity is that the regular neighbourhood is defined in terms of enclosing. Enclosing, like compatibility of splittings, is more rigid than *domination*, which is the basis for usual JSJ theory. For instance, any two splittings in Culler-Vogtmann's outer space dominate each other, but they are compatible if and only if they lie in a common simplex.

On the other hand, we have described a general construction producing a canonical splitting T_c from a canonical deformation space: the *tree of cylinders* [GL08]. It also enjoys strong compatibility properties. In the present paper we show that the splitting constructed by Scott and Swarup is a subdivision of the tree of cylinders of the usual JSJ deformation space.

More precisely, let T_J be the Bass-Serre tree of a JSJ splitting of G over VPC_n groups, as constructed for instance in [DS99]. To construct the tree of cylinders, say that two edges are in the same cylinder if their stabilizers are commensurable. Cylinders are subtrees, and the tree T_c dual to the covering of T_J by cylinders is *the tree of cylinders* of T_J (see [GL08], or Subsection 2.2 below).

Theorem 4.1. *Let G be a finitely presented group, and $n \geq 1$. Assume that G does not split over a VPC_{n-1} subgroup, and that G is not VPC_{n+1}. Let T_J be a JSJ tree of G over VPC_n subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then the Bass-Serre tree of Scott and Swarup's regular neighbourhood of all almost invariant subsets over VPC_n subgroups is equivariantly isomorphic to a subdivision of T_c.

In particular, this gives a new proof of the fact that this regular neighbourhood is a tree. Deriving the regular neighbourhood from a JSJ splitting, instead of building it from an abstract betweenness relation, seems to greatly simplify the construction; in particular this completely avoids the notion of good or good-enough position for almost invariant subsets.

There are two ingredients in our approach, to be found in Sections 3 and 4 respectively (Section 2 recalls basic material about trees of cylinders, almost invariant sets, cross-connected components, regular neighborhoods).

The first ingredient is a general fact about almost invariant sets that are *based* on a given tree T. Consider any simplicial tree T with an action of G. Any edge e separates T into two half-trees, and this defines almost invariant sets Z_e and Z_e^* (see Subsection 3.1). The collection $\mathcal{B}(T)$ of almost invariant subsets *based* on T is then defined by taking Boolean combinations of such sets Z_e.

Following Scott-Swarup, one defines cross-connected components of $\mathcal{B}(T)$ by using *crossing* of almost invariant sets. The set of cross-connected components is then endowed with a betweenness relation which allows one to constructs a bipartite graph $RN(\mathcal{B}(T))$ associated to $\mathcal{B}(T)$. This is the *regular neighborhood* of $\mathcal{B}(T)$ (see Definition 2.2).

Theorem 3.3. *Let G be a finitely generated group, and T a tree with a minimal action of G. Assume that no two groups commensurable to edge stabilizers are contained in each other with infinite index.*

Then the regular neighbourhood $RN(\mathcal{B}(T))$ is equivariantly isomorphic to a subdivision of T_c, the tree of cylinders of T for the commensurability relation; in particular, $RN(\mathcal{B}(T))$ is a tree.

The hypothesis about edge stabilizers holds in particular if all edge stabilizers of T are VPC_n for a fixed n.

This theorem remains true if one enlarges $\mathcal{B}(T)$ to $\mathcal{B}(T) \cup QH(T)$, by including almost invariant sets enclosed by quadratically hanging vertices of T. Geometrically, such a vertex is associated to a fiber bundle over a 2-dimensional orbifold $\mathcal{O}$. Any simple closed curve on $\mathcal{O}$ gives a way to blow up T by creating new edges and therefore new almost invariant sets. These sets are in $QH(T)$, as well as those associated to immersed curves on $\mathcal{O}$. Under the same hypotheses as Theorem 3.3, we show (Theorem 3.11) that the regular neighbourhood $RN(\mathcal{B}(T) \cup QH(T))$ also is a subdivision of T_c.

The second ingredient, specific to the VPC_n case, is due to (but not explicitly stated by) Scott and Swarup [SS03]. We believe it is worth emphasizing this statement, as it gives a very useful description of almost invariant sets over VPC_n subgroups in terms of a JSJ splitting T_J: in plain words, it says that any almost invariant set over a VPC_n subgroup is either dual to a curve in a QH subgroup, or is a Boolean combination of almost invariant sets dual to half-trees of T_J.

Theorem 4.2 ([DS00],[SS03, Th. 8.2])**.** *Let G and T_J be as in Theorem 4.1. For any almost invariant subset X over a VPC_n subgroup, the equivalence class $[X]$ belongs to $\mathcal{B}(T_J) \cup \mathrm{QH}(T_J)$.*

This theorem is essentially another point of view on the proof of Theorem 8.2 in [SS03] (see [SS04] for corrections), and makes a crucial use of algebraic torus theorems [DS00, DR93]. We give a proof in Section 4.

Theorem 4.1 is a direct consequence of Theorems 4.2 and 3.11.

2 Preliminaries

In this paper, G will be a fixed finitely generated group. In Section 4 it will have to be finitely presented.

2.1 Trees

If Γ is a graph, we denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of (closed) non-oriented edges.

A tree always means a simplicial tree T on which G acts without inversions. Given a family $\mathcal{E}$ of subgroups of G, an *$\mathcal{E}$-tree* is a tree whose edge stabilizers belong to $\mathcal{E}$. We denote by G_v or G_e the stabilizer of a vertex v or an edge e.

Given a subtree A, we denote by pr_A the projection onto A, mapping x to the point of A closest to x. If A and B are disjoint, or intersect in at most one point, then $pr_A(B)$ is a single point, and we define the *bridge* between A and B as the segment joining $pr_A(B)$ to $pr_B(A)$.

A tree T is *non-trivial* if there is no global fixed point, *minimal* if there is no proper G-invariant subtree.

An element or a subgroup of G is *elliptic* in T if it has a global fixed point. An element which is not elliptic is *hyperbolic*. It has an axis on which it acts as a translation. If T is minimal, then it is the union of all translation axes of elements of G. In particular, if $Y \subset T$ is a subtree, then any connected component of $T \setminus Y$ is unbounded.

A subgroup A consisting only of elliptic elements fixes a point if it is finitely generated, a point or an end in general. If a finitely generated subgroup A is not elliptic, there is a unique minimal A-invariant subtree.

A tree T *dominates* a tree T' if there is an equivariant map $f : T \to T'$. Equivalently, any subgroup which is elliptic in T is also elliptic in T'. Having the same elliptic subgroups is an equivalence relation on the set of trees, the equivalence classes are called *deformation spaces* (see [For02, GL07] for more details).

2.2 Trees of cylinders

Two subgroups A and B of G are *commensurable* if $A \cap B$ has finite index in both A and B.

Definition 2.1. *We fix a conjugacy-invariant family $\mathcal{E}$ of subgroups of G such that:*

- *any subgroup A commensurable with some $B \in \mathcal{E}$ lies in $\mathcal{E}$;*
- *if $A, B \in \mathcal{E}$ are such that $A \subset B$, then $[B : A] < \infty$.*

An $\mathcal{E}$-tree is a tree whose edge stabilizers belong to $\mathcal{E}$.

For instance, $\mathcal{E}$ may consist of all subgroups of G which are virtually $\mathbb{Z}^n$ for some fixed n, or all subgroups which are virtually polycyclic of Hirsch length exactly n.

In [GL08] we have associated a tree of cylinders T_c to any $\mathcal{E}$-tree T, as follows. Two (non-oriented) edges of T are equivalent if their stabilizers are commensurable. A *cylinder* of T is an equivalence class Y. We identify Y with the union of its edges, which is a subtree of T.

Two distinct cylinders meet in at most one point. One can then define the tree of cylinders of T as the tree T_c dual to the covering of T by its cylinders, as in [Gui04, Definition 4.8]. Formally, T_c is the bipartite tree with vertex set $V(T_c) = V_0(T_c) \sqcup V_1(T_c)$ defined as follows:

1. $V_0(T_c)$ is the set of vertices x of T belonging to (at least) two distinct cylinders;
2. $V_1(T_c)$ is the set of cylinders Y of T;
3. there is an edge $\varepsilon = (x, Y)$ between $x \in V_0(T_c)$ and $Y \in V_1(T_c)$ if and only if x (viewed as a vertex of T) belongs to Y (viewed as a subtree of T).

Alternatively, one can define the *boundary* ∂Y of a cylinder Y as the set of vertices of Y belonging to another cylinder, and obtain T_c from T by replacing each cylinder by the cone on its boundary.

All edges of a cylinder Y have commensurable stabilizers, and we denote by $\mathcal{C} \subset \mathcal{E}$ the corresponding commensurability class. We sometimes view $V_1(T_c)$ as a set of commensurability classes.

2.3 Almost invariant subsets

Given a subgroup $H \subset G$, consider the action of H on G by left multiplication. A subset $X \subset G$ is *H-finite* if it is contained in the union of finitely many H-orbits. Two subsets X, Y are *equivalent* if their symmetric difference $X + Y$ is H-finite. We denote by $[X]$ the equivalence class of X, and by X^* the complement of X.

An *H-almost invariant subset* (or an almost invariant subset over H) is a subset $X \subset G$ which is invariant under the (left) action of H, and such that, for all $s \in G$, the right-translate Xs is equivalent to X. An H-almost invariant subset X is *non-trivial* if neither X nor its complement X^* is H-finite. Given $H < G$, the set of equivalence classes of H-almost invariant subsets is a *Boolean algebra $\mathcal{B}_H$* for the usual operations.

If H contains H' with finite index, then any H-almost invariant subset X is also H'-almost invariant. Furthermore, two sets X, Y are equivalent over H' if and only if they are equivalent over H. It follows that, given a commensurability class $\mathcal{C}$ of subgroups of G, the set of equivalence classes of almost invariant subsets over subgroups in $\mathcal{C}$ is a *Boolean algebra $\mathcal{B}_{\mathcal{C}}$*.

Two almost invariant subsets X over H, and Y over K, are *equivalent* if their symmetric difference $X + Y$ is H-finite. By [SS03, Remark 2.9], this is a symmetric relation: $X + Y$ is H-finite if and only if it is K-finite. If X and Y are non-trivial, equivalence implies that H and K are commensurable.

The algebras $\mathcal{B}_{\mathcal{C}}$ are thus disjoint, except for the (trivial) equivalence classes of $\emptyset$ and G which belong to every $\mathcal{B}_{\mathcal{C}}$. We denote by $\mathcal{B}$ the union of the algebras $\mathcal{B}_{\mathcal{C}}$. It is the set of equivalence classes of all almost invariant sets, but it is not a Boolean algebra in general. There is a natural action of G on $\mathcal{B}$ induced by left translation (or conjugation).

2.4 Cross-connected components and regular neighbourhoods [SS03]

Let X be an H-almost invariant subset, and Y a K-almost invariant subset. One says that X *crosses* Y, or the pair $\{X, X^*\}$ crosses $\{Y, Y^*\}$, if none of the four sets $X^{(*)} \cap Y^{(*)}$ is H-finite (the notation $X^{(*)} \cap Y^{(*)}$ is a shortcut to denote the four possible intersections $X \cap Y$, $X^* \cap Y$, $X \cap Y^*$, and $X^* \cap Y^*$). By [Sco98], this is a symmetric relation. Note that X and Y do not cross if they are equivalent, and that crossing depends only on the equivalence classes of X and Y. Following [SS03], we will say that $X^{(*)} \cap Y^{(*)}$ is *small* if it is H-finite (or equivalently K-finite).

Now let $\mathcal{X}$ be a subset of $\mathcal{B}$. Let $\overline{\mathcal{X}}$ be the set of non-trivial unordered

pairs $\{[X],[X^*]\}$ for $[X] \in \mathcal{X}$. A *cross-connected component (CCC) of* $\mathcal{X}$ is an equivalence class C for the equivalence relation generated on $\overline{\mathcal{X}}$ by crossing. We often say that X, rather than $\{[X],[X^*]\}$, belongs to C, or represents C. We denote by $\mathcal{H}$ the set of cross-connected components of $\mathcal{X}$.

Given three distinct cross-connected components C_1, C_2, C_3, say that C_2 is *between* C_1 and C_3 if there are representatives X_i of C_i satisfying $X_1 \subset X_2 \subset X_3$.

A *star* is a subset $\Sigma \subset \mathcal{H}$ containing at least two elements, and maximal for the following property: given $C, C' \in \Sigma$, no $C'' \in \mathcal{H}$ is between C and C'. We denote by $\mathcal{S}$ the set of stars.

Definition 2.2. *Let $\mathcal{X} \subset \mathcal{B}$ be a collection of almost invariant sets. Its* regular neighbourhood $RN(\mathcal{X})$ *is the bipartite graph whose vertex set is $\mathcal{H} \sqcup \mathcal{S}$ (a vertex is either a cross-connected component or a star), and whose edges are pairs $(C,\Sigma) \in \mathcal{H} \times \mathcal{S}$ with $C \in \Sigma$. If $\mathcal{X}$ is G-invariant, then G acts on $RN(\mathcal{X})$.*

This definition is motivated by the following remark, whose proof we leave to the reader.

Remark 2.3. Let T be any simplicial tree. Suppose that $\mathcal{H} \subset T$ meets any closed edge in a nonempty finite set. Define betweenness in $\mathcal{H}$ by $C_2 \in [C_1, C_3] \subset T$. Then the bipartite graph defined as above is isomorphic to a subdivision of T.

The fact that, in the situation of Scott-Swarup, $RN(\mathcal{X})$ is a tree is one of the main results of [SS03]. We will reprove this fact by identifying $RN(\mathcal{X})$ with a subdivision of the tree of cylinders.

3 Regular neighbourhoods as trees of cylinders

In this section we fix a family $\mathcal{E}$ as in Definition 2.1: it is stable under commensurability, and a group of $\mathcal{E}$ cannot contain another with infinite index. Let T be an $\mathcal{E}$-tree.

In the first subsection we define the set $\mathcal{B}(T)$ of almost invariant sets based on T, and we state the main result (Theorem 3.3): up to subdivision, the regular neighbourhood $RN(\mathcal{B}(T))$ of $\mathcal{B}(T)$ is the tree of cylinders T_c. In Subsection 3.2, we represent elements of $\mathcal{B}(T)$ by special subforests of T. We then study the cross-connected components of $\mathcal{B}(T)$, and we prove Theorem 3.3 in Subsection 3.4 by constructing a map Φ from the set of cross-connected components to T_c. In Subsection 3.5 we generalize Theorem 3.3

by including almost invariant sets enclosed by quadratically hanging vertices of T (see Theorem 3.11).

3.1 Almost invariant sets based on a tree

We fix a basepoint $v_0 \in V(T)$. If e is an edge of T, we denote by $\mathring{e}$ the open edge. Let T_e, T_e^* be the connected components of $T \setminus \mathring{e}$. The set of $g \in G$ such that $gv_0 \in T_e$ (resp. $gv_0 \in T_e^*$) is an almost invariant set Z_e (resp. Z_e^*) over G_e. Up to equivalence, it is independent of v_0. When we need to distinguish between Z_e and Z_e^*, we orient e and we declare that the terminal vertex of e belongs to T_e.

Now consider a cylinder $Y \subset T$ and the corresponding commensurability class $\mathcal{C}$. Any Boolean combination of Z_e's for $e \in E(Y)$ is an almost invariant set over some subgroup $H \in \mathcal{C}$.

Definition 3.1. *Given a cylinder Y, associated to a commensurability class $\mathcal{C}$, the* Boolean algebra of almost invariant subsets based on Y *is the subalgebra $\mathcal{B}_\mathcal{C}(T)$ of $\mathcal{B}_\mathcal{C}$ generated by the classes $[Z_e]$, for $e \in E(Y)$.*

The set of almost invariant subsets based on T *is the union $\mathcal{B}(T) = \bigcup_\mathcal{C} \mathcal{B}_\mathcal{C}(T)$, a subset of $\mathcal{B} = \bigcup_\mathcal{C} \mathcal{B}_\mathcal{C}$ (just like $\mathcal{B}$, it is a union of Boolean algebras but not itself a Boolean algebra).*

Proposition 3.2. *Let T and T' be minimal $\mathcal{E}$-trees. Then $\mathcal{B}(T) = \mathcal{B}(T')$ if and only if T and T' belong to the same deformation space.*

More precisely, T dominates T' if and only if $\mathcal{B}(T') \subset \mathcal{B}(T)$.

Proof. Assume that T dominates T'. After subdividing T (this does not change $\mathcal{B}(T)$), we may assume that there is an equivariant map $f : T \to T'$ sending every edge to a vertex or an edge. We claim that, given $e' \in E(T')$, there are only finitely many edges $e_i \in E(T)$ such that $f(e_i) = e'$. To see this, we may restrict to a G-orbit of edges of T, since there are finitely many such orbits. If e and ge both map onto e', then $g \in G_{e'}$. Because of the hypotheses on $\mathcal{E}$, the stabilizer G_e is contained in $G_{e'}$ *with finite index*. The claim follows.

Choose basepoints $v \in T$ and $v' = f(v) \in T'$. Then $Z_{e'}$ (defined using v') is a Boolean combination of the sets Z_{e_i} (defined using v), so $\mathcal{B}(T') \subset \mathcal{B}(T)$.

Conversely, assume $\mathcal{B}(T') \subset \mathcal{B}(T)$. Let $K \subset G$ be a subgroup elliptic in T. We show that it is also elliptic in T'.

If not, we can find an edge $e' = [v', w'] \subset T'$, and sequences $g_n \in G$ and $k_n \in K$, such that the sequences $g_n v'$ and $g_n k_n v'$ have no bounded subsequence, and $e' \subset [g_n v', g_n k_n v']$ for all n (if K contains a hyperbolic

element k, we choose e' on its axis, and we define $g_n = k^{-n}$, $k_n = k^{2n}$; if K fixes an end ω, we want $g_n^{-1}e' \subset [v', k_n v']$ so we choose e' and g_n such that all edges $g_n^{-1}e'$ are contained on a ray ρ going out to ω, and then we choose k_n). Defining $Z_{e'}$ using the vertex v' and a suitable orientation of e', we have $g_n \in Z_{e'}$ and $g_n k_n \notin Z_{e'}$.

Using a vertex of T fixed by K to define the almost invariant sets Z_e, we see that any element of $\mathcal{B}(T)$ is represented by an almost invariant set X satisfying $XK = X$. In particular, since $\mathcal{B}(T') \subset \mathcal{B}(T)$, there exist finite sets F_1, F_2 such that $Z = (Z_{e'} \setminus G_{e'}F_1) \cup G_{e'}F_2$ is K-invariant on the right. For every n one has $g_n k_n \in G_{e'}F_2$ (if $g_n, g_n k_n \in Z$) or $g_n \in G_{e'}F_1$ (if not), so one of the sequences $g_n k_n v'$ or $g_n v'$ has a bounded subsequence (because $G_{e'}$ is elliptic), a contradiction. □

Remark. The only fact used in the proof is that no edge stabilizer of T has infinite index in an edge stabilizer of T'.

We can now state:

Theorem 3.3. *Let T be a minimal $\mathcal{E}$-tree, with $\mathcal{E}$ as in Definition 2.1, and T_c its tree of cylinders for the commensurability relation. Let $\mathcal{X} = \mathcal{B}(T)$ be the set of almost invariant subsets based on T.*

Then $RN(\mathcal{X})$ is equivariantly isomorphic to a subdivision of T_c.

Note that $RN(\mathcal{X})$ and T_c only depend on the deformation space of T (Proposition 3.2, and [GL08, Theorem 1]).

To prove the version of Theorem 3.3 stated in the introduction, one takes $\mathcal{E}$ to be the family of subgroups commensurable to an edge stabilizer of T.

The theorem will be proved in the next three subsections. We always fix a base vertex $v_0 \in T$.

3.2 Special forests

Let S, S' be subsets of $V(T)$. We say that S and S' are *equivalent* if their symmetric difference is finite, that S is trivial if it is equivalent to $\emptyset$ or $V(T)$.

The *coboundary* δS is the set of edges having one endpoint in S and one in S^* (the complement of S in $V(T)$). We shall be interested in sets S with finite coboundary. Since $\delta(S \cap S') \subset \delta S \cup \delta S'$, they form a Boolean algebra.

We also view such an S as a *subforest* of T, by including all edges whose endpoints are both in S; we can then consider the (connected) *components* of S. The set of edges of T is partitioned into edges in S, edges in S^*, and edges in $\delta S = \delta S^*$. Note that S is equivalent to the set of endpoints of its

edges. In particular, S is finite (as a set of vertices) if and only if it contains finitely many edges.

We say that S is a *special forest* based on a cylinder Y if $\delta S = \{e_1, \dots, e_n\}$ is finite and contained in Y. Note that S, if non-empty, contains at least one vertex of Y. Each component of S (viewed as a subforest) is a component of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$, and S^* is the union of the other components of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$.

We define $\mathcal{B}_Y$ as the Boolean algebra of equivalence classes of special forests based on Y.

Given a special forest S based on Y, we define $X_S = \{g \mid gv_0 \in S\}$. It is an almost invariant set over $H = \cap_{e \in \delta S} G_e$, a subgroup of G belonging to the commensurability class $\mathcal{C}$ associated to Y; we denote its equivalence class by $[X_S]$. Every element of $\mathcal{B}(T)$ may be represented in this form. More precisely:

Lemma 3.4. *Let Y be a cylinder, associated to a commensurability class $\mathcal{C}$. The map $S \mapsto [X_S]$ induces an isomorphism of Boolean algebras between $\mathcal{B}_Y$ and $\mathcal{B}_{\mathcal{C}}(T)$.*

Proof. It is easy to check that $S \mapsto [X_S]$ is a morphism of Boolean algebras. It is onto because the set T_e used to define the almost invariant set Z_e is a special forest (based on the cylinder containing e). There remains to determine the "kernel", namely to show that X_S is H-finite if and only if S is finite (where H denotes any group in $\mathcal{C}$).

First suppose that S is finite. Then S is contained in Y since it contains any connected component of $T \setminus Y$ which it intersects. Since δS is finite, no vertex x of S has infinite valence in T. In particular, for each vertex $x \in S$, the group G_x is commensurable with H. It follows that $\{g \in G \mid g.v_0 = x\}$ is H-finite, and X_S is H-finite.

If S is infinite, one of its components is infinite, and by minimality of T there exists a hyperbolic element $g \in G$ such that $g^n v_0 \in S$ for all $n \geq 0$. Thus $g^n \in X_S$ for $n \geq 0$. If X_S is H-finite, one can find a sequence n_i going to infinity, and $h_i \in H$, such that $g^{n_i} = h_i g^{n_0}$. Since H is elliptic in T, the sequence $h_i g^{n_0} v_0$ is bounded, a contradiction. ☐

Lemma 3.5. *Let S, S' be special forests.*

1. *If S, S' are infinite and based on distinct cylinders, and if $S \cap S'$ is finite, then $S \cap S' = \emptyset$.*

2. *If X_S crosses $X_{S'}$, then S and S' are based on the same cylinder.*

3. $X_S \cap X_{S'}$ is small if and only if $S \cap S'$ is finite.

Proof. Assume that S, S' are infinite and based on $Y \neq Y'$, and that $S \cap S'$ is finite. Let $[u, u']$ be the bridge between Y and Y' (with $u = u'$ if Y and Y' intersect in a point). Since u and u' lie in more than one cylinder, they have infinite valence in T.

Assume first that $u \in S$. Then S contains all components of $T \setminus \{u\}$, except finitely many of them (which intersect Y). In particular, S contains Y'. If S' contains u', it contains u by the same argument, and $S \cap S'$ contains infinitely many edges incident on u, a contradiction. If S' does not contain u', it is contained in S, also a contradiction.

We may therefore assume $u \notin S$ and $u' \notin S'$. It follows that S (resp. S') is contained in the union of the components of $T \setminus \{u\}$ (resp. $T \setminus \{u'\}$) which intersect Y (resp. Y'), so S and S' are disjoint. This proves Assertion 1.

Assertion 2 may be viewed as a consequence of [SS03, Prop. 13.5]. Here is a direct argument. Assume that S and S' are based on $Y \neq Y'$, and let $[u, u']$ be as above. Up to replacing S and S' by their complement, we have $u \notin S$ and $u' \notin S'$. The argument above shows that $S \cap S' = \emptyset$, so X_S does not cross $X_{S'}$.

For Assertion 3, first suppose that $S \cap S'$ is finite. If, say, S is finite, then X_S is H-finite by Lemma 3.4, so $X_S \cap X_{S'}$ is small. Assume therefore that S and S' are infinite. If they are based on distinct cylinders, then $X_S \cap X_{S'} = \emptyset$ by Assertion 1. If they are based on the same cylinder, then $S \cap S'$ is itself a finite special forest, so $X_S \cap X_{S'} = X_{S \cap S'}$ is small by Lemma 3.4. Conversely, if $S \cap S'$ is infinite, one shows that $X_S \cap X_{S'}$ is not H-finite as in the proof of Lemma 3.4, using g such that $g^n v \in S \cap S'$ for all $n \geq 0$. $\square$

Remark 3.6. If S, S' are infinite and $X_S \cap X_{S'}$ is small, then S and S' are equivalent to disjoint special forests. This follows from the lemma if they are based on distinct cylinders. If not, one replaces S' by $S' \cap S^*$.

3.3 Peripheral cross-connected components

Theorem 3.3 is trivial if T is a line, so we can assume that each vertex of T has valence at least 3 (we now allow G to act with inversions). We need to understand cross-connected components. By Assertion 2 of Lemma 3.5, every cross-connected component is based on a cylinder, so we focus on a given Y. We first define *peripheral* special forests and almost invariant sets.

Recall that ∂Y is the set of vertices of Y which belong to another cylinder. Suppose $v \in \partial Y$ is a vertex whose valence in Y is finite. Let $e_1, \dots, e_n$ be the edges of Y which contain v, oriented towards v. Let

$S_{v,Y} = T_{e_1} \cap \cdots \cap T_{e_n}$ (recall that T_e denotes the component of $T \setminus \mathring{e}$ which contains the terminal point of e). It is a subtree satisfying $S_{v,Y} \cap Y = \{v\}$, with coboundary $\delta S_{v,Y} = \{e_1, \dots, e_n\}$. We say that $S_{v,Y}$, and any special forest equivalent to it, is *peripheral* (but $S^*_{v,Y}$ is not peripheral in general).

We denote by $X_{v,Y}$ the almost invariant set corresponding to $S_{v,Y}$, and we say that X is *peripheral* if it is equivalent to some $X_{v,Y}$. Both $S_{v,Y}$ and $S^*_{v,Y}$ are infinite, so $X_{v,Y}$ is non-trivial by Lemma 3.4.

We claim that $C_{v,Y} = \{\{[X_{v,Y}], [X^*_{v,Y}]\}\}$ is a complete cross-connected component of $\mathcal{B}(T)$, called a *peripheral* CCC. Indeed, assume that $X_{v,Y}$ crosses some X_S. Then S is based on Y by Lemma 3.5, but since $S_{v,Y}$ contains no edge of Y it is contained in S_X or S^*_X, which prevents crossing.

Note that if $C_{v,Y} = C_{v',Y'}$ then $Y = Y'$ (because an H-almost invariant subset determines the commensurability class of H), and $v = v'$ except when Y is a single edge vv' in which case $X_{v,Y} = X^*_{v',Y}$.

Lemma 3.7. *Let Y be a cylinder. There is at most one non-peripheral cross-connected component C_Y based on Y. There is one if and only if $|\partial Y| \neq 2, 3$.*

Proof. We divide the proof into three steps.

• We first claim that, given any infinite connected non-peripheral special forest S based on Y, there is an edge $e \subset S \cap Y$ such that both connected components of $S \setminus \{\mathring{e}\}$ are infinite.

Assume there is no such e. Then $S \cap Y$ is locally finite: given $v \in S$, all but finitely many components of $S \setminus \{v\}$ are infinite, so infinitely many edges incident on v satisfy the claim if v has infinite valence in $S \cap Y$.

Since S is infinite and non-peripheral, $S \cap Y$ is not reduced to a single point. We orient every edge e of $S \cap Y$ so that $S \cap T_e$ is infinite and $S \cap T^*_e$ is finite. If a vertex v of $S \cap Y$ is terminal in every edge of $S \cap Y$ that contains it, S is peripheral. We may therefore find an infinite ray $\rho \subset S \cap Y$ consisting of positively oriented edges. Since every vertex of T has valence ≥ 3, every vertex of ρ is the projection onto ρ of an edge of δS, contradicting the finiteness of δS. This proves the claim.

• To show that there is at most one non-peripheral cross-connected component, we fix two non-trivial forests S, S' based on Y and we show that X_S and $X_{S'}$ are in the same CCC, provided that they do not belong to peripheral CCC's. We can assume that $X_S \cap X_{S'}$ is small, and by Remark 3.6 that $S \cap S'$ is empty. We may also assume that every component of S and S' is infinite.

Since S is not peripheral, it contains two disjoint infinite special forests S_1, S_2 based on Y: this is clear if S has several components, and follows from

the claim otherwise. Construct S_1', S_2' similarly. Then $X_{S_1} \cup X_{S_1'}$ crosses both X_S and $X_{S'}$, so X_S and $X_{S'}$ are in the same cross-connected component.

• Having proved the uniqueness of C_Y, we now discuss its existence. If $|\partial Y| \geq 4$, choose $v_1, \dots, v_4 \in \partial Y$, and consider edges e_1, e_2, e_3 of Y such that each v_i belongs to a different component S_i of $T \setminus \{\mathring{e}_1, \mathring{e}_2, \mathring{e}_3\}$. These components are infinite because $v_i \in \partial Y$, and $X_{S_1 \cup S_2}$ belongs to a non-peripheral CCC.

If $\partial Y = \emptyset$, then $Y = T$ and existence is clear. If ∂Y is non-empty, minimality of T implies that Y is the convex hull of ∂Y (replacing every cylinder by the convex hull of its boundary yields an invariant subtree). From this one deduces that $|\partial Y| \neq 1$, and every CCC based on Y is peripheral if $|\partial Y|$ equals 2 or 3. There is one peripheral CCC if $|\partial Y| = 2$ (i. e. Y is a single edge), three if $|\partial Y| = 3$. □

Remark 3.8. The proof shows that, if $|\partial Y| \geq 4$, then for all $u \neq v$ in ∂Y the non-peripheral CCC is represented by a special forest S such that $u \in S$ and $v \in S^*$.

3.4 Proof of Theorem 3.3

From now on we assume that T has more than one cylinder: otherwise there is exactly one cross-connected component, and both $RN(\mathcal{X})$ and T_c are points.

It will be helpful to distinguish between a cylinder $Y \subset T$ or a point $\eta \in \partial Y$, and the corresponding vertex of T_c. We therefore denote by Y_c or η_c the vertex of T_c corresponding to Y or η.

Recall that $\mathcal{H}$ denotes the set of cross-connected components of $\mathcal{X} = \mathcal{B}(T)$. Consider the map $\Phi : \mathcal{H} \to T_c$ defined as follows:

- if $C = C_Y$ is a non-peripheral CCC, then $\Phi(C) = Y_c \in V_1(T_c)$;
- if $C = C_{v,Y}$ is peripheral, and $\#\partial Y \geq 3$, then $\Phi(C)$ is the midpoint of the edge $\varepsilon = (v_c, Y_c)$ of T_c;
- if $\#\partial Y = 2$, and C is the peripheral CCC based on Y, then $\Phi(C) = Y_c$.

In all cases, the distance between $\Phi(C)$ and Y_c is at most 1/2. If C is peripheral, $\Phi(C)$ has valence 2 in T_c.

Clearly, Φ is one-to-one. By Remark 2.3, it now suffices to show that the image of Φ meets every closed edge, and Φ preserves betweenness: given $C_1, C_2, C_3 \in \mathcal{H}$, then C_2 is between C_1 and C_3 if and only if $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$.

The first fact is clear, because $\Phi(\mathcal{H})$ contains all vertices $Y_c \in V_1(T_c)$ with $|\partial Y| \neq 3$, and the three points at distance $1/2$ from Y_c if $|\partial Y| = 3$. To control betweenness, we need a couple of technical lemmas.

If S is a non-trivial special forest, we denote by $[[S]]$ the cross-connected component represented by the almost invariant set X_S.

Let $Y \subset T$ be a cylinder. We denote by $pr_Y : T \to Y$ the projection. If Y' is another cylinder, then $pr_Y(Y')$ is a single point. This point belongs to two cylinders, hence defines a vertex of $V_0(T_c)$ which is at distance 1 from Y_c on the segment of T_c joining Y_c to Y'_c.

Let Y be a cylinder with $|\partial Y| \geq 4$. For each non-trivial special forest S' which is either based on some $Y' \neq Y$, or based on Y and peripheral, we define a point $\eta_Y(S') \in Y \subset T$ as follows. If S' is based on $Y' \neq Y$, we define $\eta_Y(S')$ to be $pr_Y(Y')$. If S' is equivalent to some $S_{v,Y}$, we define $\eta_Y(S') = v$; note that in this case $\eta_Y(S'^*)$ is not defined.

Lemma 3.9. *Let Y be a cylinder with $|\partial Y| \geq 4$. Consider two non-trivial special forests S, S' with $[[S']] \neq C_Y$ and $[[S]] = C_Y$, and assume $S' \subset S$.*

Then $\eta = \eta_Y(S') \in Y$ is defined, $\eta \in S$, and S' contains an equivalent subforest S'' with $S'' \subset pr_Y^{-1}(\{\eta\}) \subset S$.

Moreover, $\Phi([[S']])$ lies in the connected component of $T_c \setminus \{Y_c\}$ containing η_c.

Proof. Let Y' be the cylinder on which S' is based.

If $Y' = Y$, then S'^* is not peripheral, so S' is peripheral. Thus η is defined, and S' is equivalent to its subforest $S'' = S_{Y,\eta}$. Then $S'' = pr_Y^{-1}(\{\eta\}) \subset S$. In this case $\Phi([[S']])$ is the midpoint of the edge (η_c, Y_c) of T_c.

Assume that $Y' \neq Y$. Then $\eta = pr_Y(Y') \in S$: otherwise Y' would be disjoint from S, hence from S', a contradiction. It follows that $pr_Y^{-1}(\{\eta\}) \subset S$. If $\eta \in S'$, then S' contains the complement of $pr_Y^{-1}(\{\eta\})$, so $S = T$, a contradiction. Thus $\eta \notin S'$ and therefore $S' \subset pr_Y^{-1}(\{\eta\})$. The "moreover" is clear in this case since η_c is between Y_c and Y'_c, and $\Phi([[S']])$ is at distance $\leq 1/2$ from Y'_c. ◻

Lemma 3.10. *Let $S = S_{Y,u}$ be peripheral, and let S' be a non-trivial special forest with $[[S']] \neq [[S]]$. Recall that u_c is the vertex of T_c associated to u.*

1. *If $S' \subset S$, then $\Phi([[S']])$ belongs to the component of $T_c \setminus \{\Phi([[S]])\}$ which contains u_c.*

2. *If $S \subset S'$, then $\Phi([[S']])$ belongs to the component of $T_c \setminus \{\Phi([[S]])\}$ which does not contain u_c.*

Proof. If $S' \subset S$, then S' is based on some $Y' \neq Y$. Since $S' \subset S = pr_Y^{-1}(\{u\})$, we have $Y' \subset pr_Y^{-1}(\{u\})$ and u_c is between Y_c and Y'_c in T_c. The result follows since $\Phi([[S]])$ is 1/2-close to Y_c and $\Phi([[S']])$ is 1/2-close to Y'_c.

If $S \subset S'$ and $Y \neq Y'$, we have $pr_Y(Y') \neq u$ because $S' \neq T$, and the lemma follows. If $Y = Y'$, the lemma is immediate. $\square$

We can now show that Φ preserves betweenness. Consider three distinct cross-connected components $C_1, C_2, C_3 \in \mathcal{H}$. Let Y_2 be the cylinder on which C_2 is based. Note that $|\partial Y_2| \geq 4$ if C_2 is non-peripheral.

First assume that C_2 is between C_1 and C_3. By definition, there exist almost invariant subsets X_i representing C_i such that $X_1 \subset X_2 \subset X_3$. By Lemma 3.4, one can find special forests S_i with $[X_{S_i}] = [X_i]$. By Remark 3.6, since the C_i's are distinct, one can assume $S_1 \subset S_2 \subset S_3$ (if necessary, replace S_2 by $S_2 \cap S_3$, then S_1 by $S_1 \cap S_2 \cap S_3$).

If S_2 is peripheral, then $\Phi(C_1)$ and $\Phi(C_3)$ are in distinct components of $T_c \setminus \{\Phi(C_2)\}$ by Lemma 3.10, so $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$. If S_2^* is peripheral, we apply the same argument using $S_3^* \subset S_2^* \subset S_1^*$.

Assume therefore that C_2 is non-peripheral. By Lemma 3.9, the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3^*)$ are defined, and $\eta_1 \in S_2$ and $\eta_3 \in S_2^*$. In particular, $\eta_1 \neq \eta_3$. By the "moreover" we get $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$ since $\Phi(C_2) = (Y_2)_c$.

Now assume that C_2 is not between C_1 and C_3, and choose S_i with $[[S_i]] = C_i$. By Remark 3.6, we may assume that for each $i \in \{1, 3\}$ some inclusion $S_i^{(*)} \subset S_2^{(*)}$ holds. Since C_2 is not between C_1 and C_3, we may assume after changing S_i to S_i^* if needed that $S_1 \subset S_2$ and $S_3 \subset S_2$.

If S_2 or S_2^* is peripheral, Lemma 3.10 implies that $\Phi(C_1)$ and $\Phi(C_3)$ lie in the same connected component of $T_c \setminus \{\Phi(C_2)\}$, so $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$.

Assume therefore that C_2 is non-peripheral. By Lemma 3.9, the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3)$ are defined, and we may assume $S_i \subset pr_{Y_2}^{-1}(\{\eta_i\})$. If $\eta_1 = \eta_3$, then $\Phi(C_2)$ does not lie between $\Phi(C_1)$ and $\Phi(C_3)$ by the "moreover" of Lemma 3.9. If $\eta_1 \neq \eta_3$, consider $\tilde{S}_2$ with $[[\tilde{S}_2]] = C_2$ such that $\eta_1 \in \tilde{S}_2$ and $\eta_3 \in \tilde{S}_2^*$ (it exists by Remark 3.8). Then $S_1 \subset pr_{Y_2}^{-1}(\eta_1) \subset \tilde{S}_2$ and $S_3 \subset pr_{Y_2}^{-1}(\eta_3) \subset \tilde{S}_2^*$ so C_2 lies between C_1 and C_3, a contradiction.

This ends the proof of Theorem 3.3.

3.5 Quadratically hanging vertices

We say that a vertex stabilizer G_v of T is a *QH-subgroup* if there is an exact sequence $1 \to F \to G_v \xrightarrow{\pi} \Sigma \to 1$, where $\Sigma = \pi_1(\mathcal{O})$ is a hyperbolic 2-

orbifold group and every incident edge group G_e is peripheral: it is contained with finite index in the preimage by π of a boundary subgroup $B = \pi_1(C)$, with C a boundary component of $\mathcal{O}$. We say that v is a *QH-vertex* of T.

We now define almost invariant sets based on v. They will be included in our description of the regular neighbourhood.

We view Σ as a convex cocompact Fuchsian group acting on $\mathbb{H}^2$. Let $\bar{H}$ be any non-peripheral maximal two-ended subgroup of Σ (represented by an immersed curve or 1-suborbifold). Let γ be the geodesic invariant by $\bar{H}$. It separates $\mathbb{H}^2$ into two half-spaces $P^{\pm}$ (which may be interchanged by certain elements of $\bar{H}$).

Let $\bar{H}_0$ be the stabilizer of P^+, which has index at most 2 in $\bar{H}$, and x_0 a basepoint. We define an $\bar{H}_0$-almost invariant set $\bar{X} \subset \Sigma$ as the set of $g \in \Sigma$ such that $gx_0 \in P^+$ (if $\bar{H}$ is the fundamental group of a two-sided simple closed curve on $\mathcal{O}$, there is a one-edge splitting of Σ over $\bar{H}$, and $\bar{X}$ is a Z_e as defined in Subsection 3.1).

The preimage of $\bar{X}$ in G_v is an almost invariant set X_v over the preimage H_0 of $\bar{H}_0$. We extend it to an almost invariant set X of G as follows. Let S' be the set of vertices $u \neq v$ of T such that, denoting by e the initial edge of the segment $[v, u]$, the geodesic of $\mathbb{H}^2$ invariant under $G_e \subset G_v$ lies in P^+ (note that it lies in either P^+ or P^-). Then X is the union of X_v with the set of $g \notin G_v$ such that $gv \in S'$.

Starting from $\bar{H}$, we have thus constructed an almost invariant set X, which is well-defined up to equivalence and complementation (because of the choices of x_0 and $P^{\pm}$). We say that X is a *QH-almost invariant subset* based on v. We let $\mathrm{QH}_v(T)$ be the set of equivalence classes of QH-almost invariant subsets obtained from v as above (varying $\bar{H}$), and $\mathrm{QH}(T)$ be the union of all $\mathrm{QH}_v(T)$ when v ranges over all QH-vertices of T.

Theorem 3.11. *With $\mathcal{E}$ and T as in Theorem 3.3, let $\hat{\mathcal{X}} = \mathcal{B}(T) \cup \mathrm{QH}(T)$. Then $RN(\hat{\mathcal{X}})$ is isomorphic to a subdivision of T_c.*

Proof. The proof is similar to that of Theorem 3.3.

If X is a QH-almost invariant subset as constructed above, we call $S = S' \cup \{v\}$ the *QH-forest* associated to X. We say that it is based on v. The coboundary of S is infinite, but all its edges contain v. We may therefore view S as a subtree of T (the union of v with certain components of $T \setminus \{v\}$). It is a union of cylinders. We let $S^* = (T \setminus S) \cup \{v\}$, so that $S \cap S^* = \{v\}$.

Note that S cannot contain a peripheral special forest $S_{v,Y}$, with Y a cylinder containing v (this is because the subgroup $\bar{H} \subset \Sigma$ was chosen non-peripheral).

Conversely, given a QH-forest S, one can recover H_0, which is the stabilizer of S, and the equivalence class of X. In other words, there is a bijection between $QH_v(T)$ and the set of QH-forests based on v. We denote by X_S the almost invariant set X corresponding to S (it is well-defined up to equivalence). Note that $X_S \subsetneq \{g \in G \mid gv \in S\}$, and these sets have the same intersection with $G \setminus G_v$.

The following fact is analogous to Lemma 3.5.

Lemma 3.12. *Let S be a QH-forest based on v. Let S' be a non-trivial special forest, or a QH-forest based on $v' \neq v$.*

1. *$X_S \cap X_{S'}$ is small if and only if $S \cap S' = \emptyset$.*

2. *X_S and $X_{S'}$ do not cross.*

Proof. When S' is a special forest, we use v as a basepoint to define $X_{S'} = \{g \mid gv \in S'\}$. Beware that X_S is properly contained in $\{g \mid gv \in S\}$.

We claim that, if S' is a special forest with $v \notin S'$ and $S \cap S' \neq \emptyset$, then $X_{S'} \subset X_S$. Indeed, let Y' be the cylinder on which S' is based. Since each connected component of S' contains a point in Y', there is a point $w \neq v$ in $S \cap Y'$. As S is a union of cylinders, S contains Y'. All connected components of S' therefore contain a point of S, so are contained in $S \setminus \{v\}$ since $v \notin S'$. We deduce $X_{S'} \subset X_S$.

We now prove assertion 1. If $S \cap S' = \emptyset$, then $X_S \cap X_{S'} = \emptyset$. We assume $S \cap S' \neq \emptyset$, and we show that $X_S \cap X_{S'}$ is not small. If S' is a QH-forest, then $v \in S'$ or $v' \in S$. If for instance $v \in S'$, then $X_S \cap X_{S'}$ is not small because it contains $X_S \cap G_v$. Now assume that S' is a special forest. If $v \in S'$, the same argument applies, so assume $v \notin S'$. The claim implies $X_{S'} \subset X_S$, so $X_S \cap X_{S'}$ is not small.

To prove 2, first consider the case where S' is a QH-forest. Up to changing S and S' to S^* or S'^*, one can assume $S \cap S' = \emptyset$ so X_S does not cross $X_{S'}$. If S' is a special forest, we can assume $v \notin S'$ by changing S' to S'^*. By the claim, X_S does not cross $X_{S'}$. $\square$

The lemma implies that no element of $\mathrm{QH}(T)$ crosses an element of $\mathcal{B}(T)$, and elements of $\mathrm{QH}_v(T)$ do not cross elements of $\mathrm{QH}_{v'}(T)$ for $v \neq v'$.

Since $\mathrm{QH}_v(T)$ is a cross-connected component, the set $\hat{\mathcal{H}}$ of cross-connected components of $\mathcal{B}(T) \cup \mathrm{QH}(T)$ is therefore the set of cross-connected components of $\mathcal{B}(T)$, together with one new cross-connected component $\mathrm{QH}_v(T)$ for each QH-vertex v.

One extends the map Φ defined in the proof of Theorem 3.3 to a map $\hat{\Phi} : \hat{\mathcal{H}} \to T_c$ by sending $\mathrm{QH}_v(T)$ to v (viewed as a vertex of $V_0(T_c)$ since a

QH-vertex belongs to infinitely many cylinders). We need to prove that $\hat{\Phi}$ preserves betweenness.

Lemmas 3.9 and 3.10 extend immediately to the case where S' is a QH-forest: one just needs to define $\eta_Y(S') = pr_Y(v')$ for S' based on v', so that v' plays the role of Y' in the proofs (in the proof of 3.9, the assertion that $\eta \notin S'$ should be replaced by the fact that $S' \cap Y$ contains no edge; this holds since otherwise S' would contain Y). This allows to prove that, if C_2 is not a component $\mathrm{QH}_v(T)$, then $\Phi(C_2)$ is between $\Phi(C_1)$ and $\Phi(C_3)$ if and only if C_2 lies between C_1 and C_3.

To deal with the case when $C_2 = \mathrm{QH}_v(T)$, we need a cylinder-valued projection η_v. Let Y be a cylinder or a QH-vertex distinct from v. We define $\eta_v(Y)$ as the cylinder of T containing the initial edge of $[v, x]$ for any $x \in Y$ different from v. Equivalently, $\eta_v(Y)$ is Y if $v \in Y$, the cylinder containing the initial edge of the bridge joining x to Y otherwise.

If v lies in a cylinder Y^0, denote by $\eta_v^{-1}(Y^0)$ the union of cylinders Y such that $\eta_v(Y) = Y^0$. Equivalently, this is the set of points $x \in T$ such that $x = v$ or $[x, v]$ contains an edge of Y^0.

As before, we denote by $[[S]]$ the cross-connected component represented by X_S.

Lemma 3.13. *Let S be a QH-forest based on v. Let S' be a non-trivial special forest, or a QH-forest based on $v' \neq v$. Let Y' be the cylinder or QH-vertex on which S' is based, and $Y'^0 = \eta_v(Y')$.*

If $S' \subset S$, then $S' \subset \eta_v^{-1}(Y'^0) \subset S$.

Moreover, $\Phi([[S']])$ and $Y_c'^0$ lie in the same component of $T_c \setminus \{\Phi([[S]])\}$.

We leave the proof of this lemma to the reader.

Assume now that $S_1 \subset S_2 \subset S_3$ with $[[S_i]] = C_i$ and S_2 based on v. For $i = 1, 3$ let $Y_i^0 = \eta_v(Y_i)$. Then $S_1 \subset \eta_v^{-1}(Y_1^0) \subset S_2$ and $S_3^* \subset \eta_v^{-1}(Y_3^0) \subset S_2^*$. In particular, $Y_1^0 \neq Y_3^0$. Since $(Y_1^0)_c$ and $(Y_3^0)_c$ are neighbours of v_c, they lie in distinct components of $T_c \setminus \{\Phi(C_2)\}$. By Lemma 3.13, so do $\Phi([[S_1]])$ and $\Phi([[S_3]])$.

Conversely, assume that C_2 does not lie between C_1 and C_3, and consider $S_1 \subset S_2$ and $S_3 \subset S_2$ with $[[S_i]] = C_i$. For $i = 1, 3$, let Y_i^0 be as above. If $Y_1^0 = Y_3^0$, then $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$ by Lemma 3.13, and we are done. If $Y_1^0 \neq Y_3^0$, these cylinders correspond to distinct peripheral subgroups of G_v, with invariant geodesics $\gamma_1 \neq \gamma_3$. There exists a non-peripheral group $\bar{H} \subset \Sigma$, as in the beginning of this subsection, whose invariant geodesic separates γ_1 and γ_3. Let S_2' be the associated QH-forest. Then $[[S_2']] = C_2$ and, up to complementation, $\eta_v^{-1}(Y_1^0) \subset S_2'$ and $\eta_v^{-1}(Y_3^0) \subset$

${S_2'}^*$. It follows that $S_1 \subset S_2'$ and $S_3^* \subset {S_2'}^*$, so C_2 lies between C_1 and C_3, contradicting our assumptions. □

4 The regular neighbourhood of Scott and Swarup

A group is VPC_n if some finite index subgroup is polycyclic of Hirsch length n. For instance, VPC_0 groups are finite groups, VPC_1 groups are virtually cyclic groups, VPC_2 groups are virtually $\mathbb{Z}^2$ (but not all VPC_n groups are virtually abelian for $n \geq 3$).

Fix $n \geq 1$. We assume that G is finitely presented and does not split over a VPC_{n-1} subgroup. We also assume that G itself is not VPC_{n+1}. All trees considered here are assumed to have VPC_n edge stabilizers.

A subgroup $H \subset G$ is *universally elliptic* if it is elliptic in every tree. A tree is universally elliptic if all its edge stabilizers are.

A tree is a *JSJ tree* (over VPC_n subgroups) if it is universally elliptic, and maximal for this property: it dominates every universally elliptic tree. JSJ trees exist (because G is finitely presented) and belong to the same deformation space, called the JSJ deformation space (see [GLa]).

A vertex stabilizer G_v of a JSJ tree is *flexible* if it is not VPC_n and is not universally elliptic. It follows from [DS99] that a flexible vertex stabilizer is a QH-subgroup (as defined in Subsection 3.5): there is an exact sequence $1 \to F \to G_v \to \Sigma \to 1$, where $\Sigma = \pi_1(\mathcal{O})$ is the fundamental group of a hyperbolic 2-orbifold, F is VPC_{n-1}, and every incident edge group G_e is peripheral. Note that the QH-almost invariant subsets X constructed in Subsection 3.5 are over VPC_n subgroups.

We can now describe the regular neighbourhood of all almost invariant subsets of G over VPC_n subgroups as a tree of cylinders.

Theorem 4.1. *Let G be a finitely presented group, and $n \geq 1$. Assume that G does not split over a* VPC_{n-1} *subgroup, and that G is not* VPC_{n+1}. *Let T be a JSJ tree over* VPC_n *subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then Scott and Swarup's regular neighbourhood of all almost invariant subsets over VPC_n *subgroups is equivariantly isomorphic to a subdivision of T_c.*

This is immediate from Theorem 3.11 and the following result saying that one can read any almost invariant set over a VPC_n subgroup in a JSJ tree T.

Theorem 4.2 ([DS00],[SS03, Th. 8.2]). *Let G and T be as above. For any almost invariant subset X over a VPC_n subgroup, the equivalence class $[X]$ belongs to $\mathcal{B}(T) \cup \mathrm{QH}(T)$.*

Proof. We essentially follow the proof by Scott and Swarup. For definitions, see [SS03]. All trees considered here have VPC_n edge stabilizers.

Let X be an almost invariant subset over a VPC_n subgroup H. We assume that it is non-trivial. We first consider the case where X crosses strongly some other almost invariant subset. Then by [DS00, Proposition 4.11] H is contained as a non-peripheral subgroup in a QH-vertex stabilizer W of some tree T'. When acting on T, the group W fixes a QH-vertex $v \in T$ (see [GLa]).

Note that H is not peripheral in G_v, because it is not peripheral in W. Since (G, H) only has 2 co-ends [SS03, Proposition 13.8], there are (up to equivalence) only two almost invariant subsets over subgroups commensurable with H (namely X and X^*), and therefore $[X] \in \mathrm{QH}_v(T)$.

¿From now on, we assume that X crosses strongly no other almost invariant subset over a VPC_n subgroup. Then, by [DR93] and [DS00, Section 3], there is a non-trivial tree T_0 with one orbit of edges and an edge stabilizer H_0 commensurable with H.

Since X crosses strongly no other almost invariant set, H and H_0 are universally elliptic (see [Gui05, Lemme 11.3]). In particular, T dominates T_0. It follows that there is an edge of T with stabilizer contained in H_0 (necessarily with finite index). This edge is contained in a cylinder Y associated to the commensurability class of H.

The main case is when T has no edge e such that Z_e crosses X (see Subsection 3.1 for the definition of Z_e). The following lemma implies that X is enclosed in some vertex v of T.

Lemma 4.3. *Let G be finitely generated. Let $X \subset G$ be a non-trivial almost invariant set over a finitely generated subgroup H. Let T be a tree with an action of G. If X crosses no Z_e, then X is enclosed in some vertex $v \in T$.*

Proof. The argument follows a part of the proof of Proposition 5.7 in [SS03, SS04].

Given two almost invariant subsets, we use the notation $X \geq Y$ when $Y \cap X^*$ is small. The non-crossing hypothesis says that each edge e of T may be oriented so that $Z_e \geq X$ or $Z_e \geq X^*$. If one can choose both orientations for some e, then X is equivalent to Z_e, so X is enclosed in both endpoints of e and we are done.

We orient each edge of T in this manner. We color the edge blue or red, according to whether $Z_e \geq X$ or $Z_e \geq X^*$. No edge can have both colors. If e is an oriented edge, and if e' lies in T_e^*, then e' is oriented towards e, so that $Z_e \subset Z_{e'}$, and e' has the same color as e. In particular, given a vertex v, either all edges containing v are oriented towards v, or there exists exactly one edge containing v and oriented away from v, and all edges containing v have the same color.

If v is as in the first case, X is enclosed in v by definition. If there is no such v, then all edges have the same color and are oriented towards an end of T. By Lemma 2.31 of [SS03], G is contained in the R-neighbourhood of X for some $R > 0$, so X is trivial, a contradiction. □

Let v be a vertex of T enclosing X. In particular $H \subset G_v$. The set $X_v = X \cap G_v$ is an H-almost invariant subset of G_v (note that G_v is finitely generated). By [SS03, Lemma 4.14], there is a subtree $S \subset T$ containing v, with $S \setminus \{v\}$ a union of components of $T \setminus \{v\}$, such that X is equivalent to $X_v \cup \{g \mid g.v \in S \setminus \{v\}\}$.

Lemma 4.4. *The H-almost invariant subset X_v of G_v is trivial.*

Proof. Otherwise, by [DR93, DS00], there is a G_v-tree T_1 with one orbit of edges and an edge stabilizer H_1 commensurable with H, and an edge $e_1 \subset T_1$, such that Z_{e_1} lies (up to equivalence) in the Boolean algebra generated by the orbit of X_v under the commensurator of H in G_v.

Note that G_e is elliptic in T_1 for each edge e of T incident to v: by symmetry of strong crossing ([SS03, Proposition 13.3]), G_e does not cross strongly any translate of X, and thus does not cross strongly Z_{e_1}, so G_e is elliptic in T_1 ([Gui05, Lemme 11.3]). This ellipticity allows us to refine T by creating new edges with stabilizer conjugate to H_1. Since H_1 is universally elliptic, this contradicts the maximality of the JSJ tree T. □

After replacing X by an equivalent almost invariant subset or its complement, and possibly changing S to $(T \setminus S) \cup \{v\}$, we can assume that $X = \{g \mid g.v \notin S\}$. Recall that Y is the cylinder defined by the commensurability class of H.

Lemma 4.5. *The coboundary δS, consisting of edges vw with $w \notin S$, is a finite set of edges of Y.*

This implies that $[X] \in \mathcal{B}(T)$, ending the proof when X crosses no Z_e.

Proof of Lemma 4.5. Let E be the set of edges of δS, oriented so that $X = \sqcup_{e\in E} Z_e$ (we use v as a basepoint to define Z_e). Let A be a finite generating system of G such that, for all $a \in A$, the open segment (av, v) does not meet the orbit of v. One can construct such a generating sytem from any finite generating system by iterating the following operation: replace $\{a\}$ by the pair $\{g, g^{-1}a\}$ if (av, v) contains some $g.v$.

Let Γ be the Cayley graph of (G, A). For any subset $Z \subset G$, denote by δZ the set of edges of Γ having one endpoint in Z and the other endpoint in $G \setminus Z$. By our choice of A, no edge joins a vertex of Z_e to a vertex of $Z_{e'}$ for $e \neq e'$. It follows that $\delta X = \sqcup_{e\in E} \delta Z_e$.

Since δX is H-finite, the set δZ_e is H-finite for each $e \in E$, and E is contained in a finite union of H-orbits. Let $e \in E$. Since δZ_e is G_e-invariant and H-finite, $G_e \cap H$ has finite index in G_e. Since G_e and H are both VPC_n, they are commensurable, so the H-orbit of e is finite. It follows that $E \subset Y$ and that E is finite. □

We now turn to the case when X crosses some Z_e's. For each $e \in E(T)$, the intersection number $i(Z_e, X)$ is finite [Sco98], which means that there are only finitely many edges e' in the orbit of e such that $Z_{e'}$ crosses X. Since T/G is finite, let $e_1, e_1^{-1}, e_2, e_2^{-1}, \dots, e_n, e_n^{-1}$ be the finite set of oriented edges e such that Z_e crosses X (we denote by $e \mapsto e^{-1}$ the orientation-reversing involution). Note that $e_i \subset Y$ by [SS03, Proposition 13.5]. Now X is a finite union of sets of the form $X' = X \cap Z_{e_1^{\pm 1}} \cap \dots \cap Z_{e_n^{\pm 1}}$. Since X' does not cross any Z_e, its equivalence class lies in $\mathcal{B}(T)$ by the argument above and so does $[X]$. □

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